MATH 2010E TUTO 9

Finding Local Extrema

Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–30.

7. $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

Ans: $\left.\begin{aligned} f_x &= 4x + 3y - 5 \\ f_y &= 3x + 8y + 2 \end{aligned}\right\}$ both exist everywhere on $\mathbb{R}^2$

Set $\nabla f = (f_x, f_y) = (0, 0)$.

Then $\begin{cases} 4x + 3y = 5 \\ 3x + 8y = -2 \end{cases}$

$\Rightarrow \quad (x, y) = (2, -1)$

So the only critical pt is $(2, -1)$

all cts on $\mathbb{R}^2$

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 8 \end{pmatrix}$$ (at $(2,-1)$ in particular)

Note $\begin{cases} f_{xx} = 4 > 0 \\ f_{xx}f_{yy} - f_{xy}^2 = 32 - 9 = 23 > 0 \end{cases}$

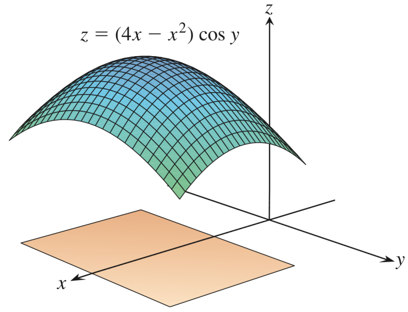
By the 2nd Derivative Test,

f attains a local minimum $f(2,-1) = -6$ at $(2,-1)$ //

Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

37. $f(x, y) = (4x - x^2)\cos y$ on the rectangular plate $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$ (see accompanying figure)



Ans: Clearly, $\Omega := \{(x,y) : 1 \le x \le 3, -\pi/4 \le y \le \pi/4\}$ is closed and bounded, and f is cts on Ω.

By EVT, f has abs max and min on Ω.

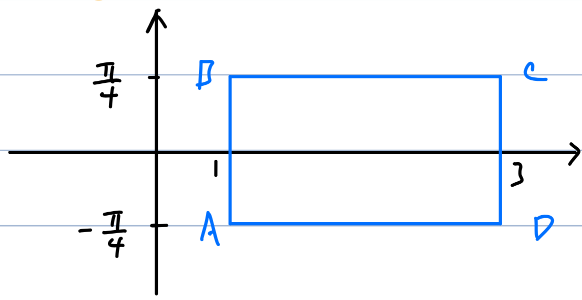
1) Critical pts in $\text{Int}(\Omega)$:

$\vec{\nabla} f = \big((4-2x)\cos y, -(4x-x^2)\sin y\big)$ exist everywhere

$$\vec{\nabla} f = \vec{0} \iff \begin{cases} (4-2x)\underbrace{\cos y}_{\neq 0} = 0 \\ \underbrace{(4x-x^2)}_{=4}\sin y = 0 \end{cases} \iff \begin{cases} x = 2 \\ y = 0 \end{cases}$$

So $(2,0)$ is the only critical pt in $\text{Int}(\Omega)$

and $f(2,0) = 4$.



2) Study f on $\partial\Omega$

- On AB, $f(x,y) = f(1,y) = 3\cos y$ for $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$

  $\frac{d}{dy} f(1,y) = -3\sin y = 0 \Rightarrow y = 0$

  $f(1,0) = 3$, $f(1, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$, $f(1, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}$,

- On BC, $f(x,y) = f(x, \frac{\pi}{4}) = \frac{1}{\sqrt{2}}(4x - x^2)$ for $1 \le x \le 3$

  $\frac{d}{dx} f(x, \frac{\pi}{4}) = \frac{1}{\sqrt{2}}(4 - 2x) = 0 \Rightarrow x = 2$

  $f(2, \frac{\pi}{4}) = \frac{4}{\sqrt{2}}$, $f(1, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}$, $f(3, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}$

- On CD, $f(x,y) = f(3,y) = 3\cos y$ for $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$

  $\frac{d}{dy} f(3,y) = -3\sin y = 0 \Rightarrow y = 0$

  $f(3,0) = 3$, $f(3,-\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$, $f(3,\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$

- On AD, $f(x,y) = f(x,-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(4x - x^2)$ for $1 \le x \le 3$

  $\frac{d}{dx} f(x,-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(4-2x) = 0 \Rightarrow x = 2$

  $f(2,-\frac{\pi}{4}) = \frac{4}{\sqrt{2}}$, $f(1,-\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$, $f(3,-\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$

3) Compare values of f at pts from 1), 2).

   abs max $= 4$ at $(2,0)$

   abs min $= \frac{3}{\sqrt{2}}$ at $(1,\frac{\pi}{4})$, $(1,-\frac{\pi}{4})$, $(3,\frac{\pi}{4})$, $(3,-\frac{\pi}{4})$ //

39. Find two numbers a and b with $a \le b$ such that

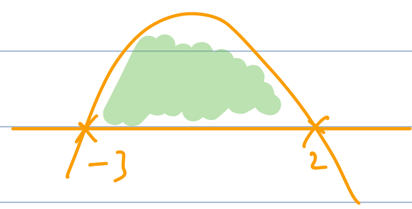
$$\int_a^b (6 - x - x^2)\, dx \qquad (\#)$$

has its largest value.

Ans: Observe $6 - x - x^2 = -(x-2)(x+3)$

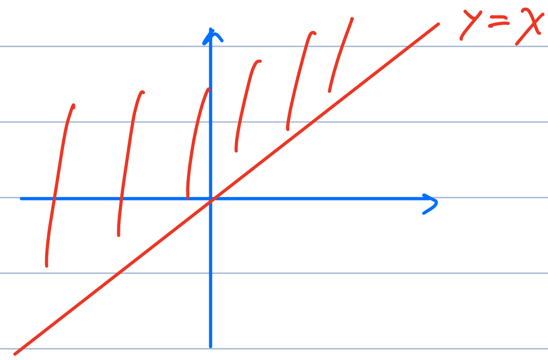
$\Rightarrow$ # is largest when

$a = -3, \quad b = 2.$



Let $F(a,b) = \int_a^b (6 - x - x^2)\,dx$

$\Omega := \{(a,b) : a \le b\}$



On $\partial\Omega$, $a = b \Rightarrow F(a,a) = 0$

For interior critical pts,

$$\begin{cases} \dfrac{\partial F}{\partial a} = -(6 - a - a^2) = 0 & \Rightarrow a = -3, 2 \\[2mm] \dfrac{\partial F}{\partial b} = (6 - b - b^2) = 0 & \Rightarrow b = -3, 2 \end{cases}$$

So, the only critical pt is $(a,b) = (-3,2)$ with $F(-3,2) = \frac{125}{6}$

Let $T = \{(a,b) : -3 \le a \le b \le 2\}$

Then $F(a,b) \le F(-3,2)$ for (a,b) on ∂T and outside T

Hence $F(a,b)$ has largest value $\frac{125}{6}$ when $(a,b) = (-3,2)$ //

53. Find three numbers whose sum is 9 and whose sum of squares is a minimum.

Ans : To minimize $f(x,y) = x^2 + y^2 + (9-x-y)^2$ for $(x,y) \in \mathbb{R}^2$

1) Note $f(x,y) \geq \|(x,y)\|^2$.

So $\lim_{\|(x,y)\| \to \infty} f(x,y) = \infty \implies$ no global max

2) Critical pts :

$\vec{\nabla} f = (2x - 2(9-x-y), 2y - 2(9-x-y)) = \vec{0}$

$\iff \begin{cases} 2x + y = 9 \\ x + 2y = 9 \end{cases} \iff (x,y) = (3,3)$

The only critical pt is $(3,3)$

Hence f has a global max at $(3,3)$ with value $f(3,3) = 27$

These 3 numbers are 3 , 3 , 3

//

Extreme Values on Parametrized Curves To find the extreme values of a function $f(x, y)$ on a curve $x = x(t)$, $y = y(t)$, we treat f as a function of the single variable t and use the Chain Rule to find where df/dt is zero. As in any other single-variable case, the extreme values of f are then found among the values at the

a. critical points (points where df/dt is zero or fails to exist), and

b. endpoints of the parameter domain.

Find the absolute maximum and minimum values of the following functions on the given curves.

61. Functions:

a. $f(x, y) = x + y$ **b.** $g(x, y) = xy$ **c.** $h(x, y) = 2x^2 + y^2$

Curves:

i) The semicircle $x^2 + y^2 = 4$, $y \geq 0$

ii) The quarter circle $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$

Use the parametric equations $x = 2\cos t$, $y = 2\sin t$.

Ans: By chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = 1\cdot(-2\sin t) + 1(2\cos t)$$

$$\frac{df}{dt} = 0 \Rightarrow \sin t = \cos t \Rightarrow x = y$$

i) On the semi-circle $x^2+y^2=4$, $y \geq 0$,

$(x,y) = (\cos t, \sin t)$, $0 \leq t \leq \pi$

$\Rightarrow$ critical pt $t = \frac{\pi}{4} \Rightarrow x = y = \sqrt{2}$, $f(\sqrt{2},\sqrt{2}) = 2\sqrt{2}$

end pts $t = 0 \Rightarrow (x,y) = (2,0)$. $f(2,0) = 2$

$t = \pi \Rightarrow (x,y) = (-2,0)$, $f(-2,0) = -2$

So abs max $= 2\sqrt{2}$

abs min $= -2$

ii) On the quarter circle $x^2+y^2=4$, $x, y \geq 0$

$(x,y) = (\cos t, \sin t)$, $0 \leq t \leq \pi/2$

$\Rightarrow$ critical pt $t = \frac{\pi}{4} \Rightarrow x = y = \sqrt{2}$, $f(\sqrt{2},\sqrt{2}) = 2\sqrt{2}$

end pts $t = 0 \Rightarrow (x,y) = (2,0)$. $f(2,0) = 2$

$t = \pi/2 \Rightarrow (x,y) = (0,2)$, $f(0,2) = 2$

So abs max $= 2\sqrt{2}$

abs min $= 2$

65. Least squares and regression lines When we try to fit a line $y = mx + b$ to a set of numerical data points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$, we usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line. In theory, this means finding the values of m and b that minimize the value of the function

$$w = (mx_1 + b - y_1)^2 + \cdots + (mx_n + b - y_n)^2. \tag{1}$$

(See the accompanying figure.) Show that the values of m and b that do this are

$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n\sum x_k y_k}{\left(\sum x_k\right)^2 - n\sum x_k{}^2}, \tag{2}$$

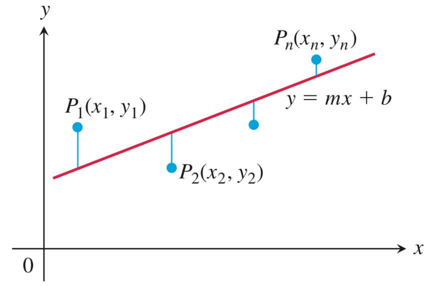
$$b = \frac{1}{n}\left(\sum y_k - m\sum x_k\right), \tag{3}$$

with all sums running from $k = 1$ to $k = n$. Many scientific calculators have these formulas built in, enabling you to find m and b with only a few keystrokes after you have entered the data.

The line $y = mx + b$ determined by these values of m and b is called the **least squares line**, **regression line**, or **trend line** for the data under study. Finding a least squares line lets you

1. summarize data with a simple expression,
2. predict values of y for other, experimentally untried values of x,
3. handle data analytically.

We demonstrated these ideas with a variety of applications in Section 1.4.



Ans: To minimize $w(m,b) = \sum_{k=1}^{n} (mx_k + b - y_k)^2$ on $\mathbb{R}^2$

Note $\lim_{\|(m,b)\| \to \infty} w(m,b) = \infty$

Consider

$$\begin{cases} \dfrac{\partial w}{\partial m} = \sum_{k=1}^{n} 2(mx_k + b - y_k)\cdot x_k = 2m\sum x_k^2 + 2b\sum x_k - 2\sum x_k y_k = 0 & ① \\ \dfrac{\partial w}{\partial b} = \sum_{k=1}^{n} 2(mx_k + b - y_k) = 2m\sum x_k + 2nb - 2\sum y_k = 0 & ② \end{cases}$$

① $\times n$ − ② $\times \sum x_k$:

$$\left(2n\sum x_k^2 - 2\left(\sum x_k\right)^2\right)m - 2n\sum x_k y_k + 2\left(\sum x_k\right)\left(\sum y_k\right) = 0$$

$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n\sum x_k y_k}{\left(\sum x_k\right)^2 - n\sum x_k^2}$$

② :

$$b = \frac{1}{n}\left(\sum y_k - m\sum x_k\right)$$

give the global min.